MEASURES ON PROJECTIONS IN A W^*-ALGEBRA OF TYPE I_2

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Abstract. It is shown that for every measure m on projections in a W^*-algebra of type I_2, there exists a Hilbert-valued orthogonal vector measure μ such that $\|\mu(p)\|^2 = m(p)$ for every projection p. With regard to J. Hamhalter's result (Proc. Amer. Math. Soc., 110 (1990), 803–806) it means that the assertion is valid for an arbitrary W^*-algebra.

It is well known that the problem of the extension of a measure on projections to a linear functional was positively solved for W^*-algebras without type I_2 direct summand. (A lucid exposition of Gleason-Christensen-Yeadon'results see in [2].) In view of this, it became a good tradition to exclude the W^*-algebras with direct summand of type I_2 when measures on projections are investigated. In this respect, there is an interesting paper by J. Hamhalter [1] which describes the connection between measures on projections in conventional sense and H-valued (H is a complex Hilbert space) orthogonal vector measures. Specifically, it has been proved in [1] (though expressed in a slightly different form) that if m is a measure on projections in a W^*-algebra $\mathcal{A}$ without type I_2 direct summand, then there exists a H-valued orthogonal vector measure μ on projections in $\mathcal{A}$ such that $\|\mu(p)\|^2 = m(p)$ for every $p \in \mathcal{A}$. The mentioned proof (in a few lines) of this assertion is based on Gleason-Christensen-Yeadon's result.

In this paper we give a construction allowing to obtain a proof of this assertion for W^*-algebras of type I_2 and therefore for arbitrary W^*-algebras. The author is greatly indebted to Lugovaya G.D. for useful discussions.

Preliminaries

Let $\mathcal{A}$ be a W^*-algebra, and $\mathcal{A}^{\rm pr}$, $\mathcal{A}^{\rm un}$, $\mathcal{A}^+$ denote the sets of orthogonal projections, unitaries, positive elements in $\mathcal{A}$, respectively. We will denote by $\mathrm{rp}\,(x)$ the range projection of $x \in \mathcal{A}^+$. It is the least projection of all projections $p \in \mathcal{A}^{\rm pr}$ such that $px = x$. It should be noted that $\mathrm{rp}\,(x) = \mathrm{rp}\,(x^{1/2})$. The basic notions those we talk about in this paper are described by the following

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definitions. (Monograph [3] gives further details of problems related to measures on projections in von Neumann algebras.)

Definition 1. Let $\mathcal{A}$ be a W^*-algebra. A mapping $m : \mathcal{A}^{\mathrm{pr}} \to \mathbb{R}_+$ is called *a measure on projections* if the following condition is satisfied:

$$p = \sum_{i \in I} p_i \ \ (p, p_i \in \mathcal{A}^{\mathrm{pr}},\ p_i p_j = 0\ (i \neq j)) \Rightarrow m(p) = \sum_{i \in I} m(p_i).$$

Here, the series are understood as limits of the nets of finite sums (in w^*-topology for projections).

Definition 2. Let $\mathcal{A}$ be a W^*-algebra, H be a complex Hilbert space. A mapping $\mu : \mathcal{A}^{\mathrm{pr}} \to H$ is called *an orthogonal vector measure* if for any set $(p_j)_{j \in J} \subset \mathcal{A}^{\mathrm{pr}}$ of mutually orthogonal projections the following two conditions are satisfied:

(i) the set $(\mu(p_j))_{j \in J}$ is orthogonal in H,

(ii) $\mu(\sum\limits_{j \in J} p_j) = \sum\limits_{j \in J} \mu(p_j)$,

where the series on the right hand side are understood as the limit of the net of finite partial sums (in the norm topology on H).

Let $X \subset \mathcal{A}^{\mathrm{pr}}$ has the property

(iii) $p, q \in X,\ pq = 0 \Rightarrow p + q \in X$.

We call $\mu : X \to H$ *a finitely additive orthogonal vector measure on* X if the following condition is satisfied

$$p, q \in X,\ pq = 0 \ \Rightarrow\ \langle \mu(p), \mu(q) \rangle = 0,\ \mu(p+q) = \mu(p) + \mu(q).$$

We are interested here in W^*-algebras of type I_2. It is known that the every W^*-algebra $\mathcal{N}$ of type I_2 can be expressed in the form $\mathcal{N} = \mathcal{M} \otimes M_2$ where $\mathcal{M}$ is a commutative W^*-algebra and M_2 is the algebra of all 2×2 matrices over $\mathbb{C}$.

We turn our attention to the structure of projections in algebra $\mathcal{N}$. We will consider projections in $\mathcal{N}^{\mathrm{pr}}$ defined as follows:

$$\pi_1 \oplus \pi_2 \equiv \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}, \qquad \pi_1, \pi_2 \in \mathcal{M}^{\mathrm{pr}},$$

$$p(x, v, \pi) \equiv \begin{pmatrix} x & v(x(\pi - x))^{1/2} \\ v^*(x(\pi - x))^{1/2} & \pi - x \end{pmatrix},$$

where $\pi \in \mathcal{M}^{\mathrm{pr}},\ v \in \mathcal{M}^{\mathrm{un}},\ 0 \leq x \leq \pi,\ \mathrm{rp}\,(x(\pi - x)) = \pi$. In particular, $p(0, v, 0) = 0$.

The following two lemmas are fairly straightforward from equalities $p = p^2 = p^*$ for a projection p.

Lemma 1. *Every projection $p \in \mathcal{N}^{\mathrm{pr}}$ can be expressed in the form:*

$$p = \pi_1 \oplus \pi_2 + p(x, v, \pi),$$

where $\pi_i \leq \mathbf{1} - \pi,\ i = 1, 2$[1].

We will denote

$$\pi \setminus \rho \equiv \pi - \pi\rho, \quad \pi\Delta\rho \equiv (\pi \setminus \rho) + (\rho \setminus \pi), \qquad \pi, \rho \in \mathcal{M}^{\mathrm{pr}}.$$

Let us observe some useful properties of the mentioned representation for projections.

Lemma 2. *$p(x, v, \pi)p(y, w, \rho) = 0$ if and only if*

$$y\pi\rho = (\mathbf{1} - x)\pi\rho, \quad w\pi\rho = -v\pi\rho.$$

In addition,

$$p(x, v, \pi) + p(y, w, \rho) = \pi\rho \oplus \pi\rho + p(z, u, \pi\Delta\rho)$$

where $z = x(\pi \setminus \rho) + y(\rho \setminus \pi)$ and $u \in \mathcal{M}^{\mathrm{un}}$ satisfies equations: $u(\pi \setminus \rho) = v(\pi \setminus \rho),\ u(\rho \setminus \pi) = w(\rho \setminus \pi)$.

Specifically, $p(x, v, \pi)p(y, w, \pi) = 0$ if and only if $y\pi = (\mathbf{1} - x)\pi, \quad w\pi = -v\pi$. In addition,

$$p(x, v, \pi) + p(\mathbf{1} - x, -v, \pi) = \pi \oplus \pi.$$

Lemma 3. *Let $\mathcal{A}$ be a W^*-algebra, $m : \mathcal{A}^{\mathrm{pr}} \to \mathbb{R}_+$ be a measure on projections and $\mu : \mathcal{A}^{\mathrm{pr}} \to H$ be a finitely additive orthogonal vector measure with*

$$\|\mu(p)\|^2 = m(p), \qquad p \in \mathcal{A}^{\mathrm{pr}}.$$

Then μ is the orthogonal vector measure.

Proof. It should be enough to verify the property (ii) in Definition 2. Let $p = \sum_{j \in J} p_j = \mathrm{w}^*\text{-}\lim_\sigma \sum_{j \in \sigma} p_j$ (the limit of the net of finite partial sums). Since (ii) is fulfilled for finite sums, we have

$$\begin{aligned} \|\mu(p) - \sum_{j \in \sigma} \mu(p_j)\|^2 = \|\mu(p - \sum_{j \in \sigma} p_j)\|^2 = m(p - \sum_{j \in \sigma} p_j) \\ = m(p) - \sum_{j \in \sigma} m(p_j). \end{aligned}$$

As m is completely additive, it follows $\lim_\sigma [m(p) - \sum_{j \in \sigma} m(p_j)] = 0$. ☐

We need also the following elementary lemma.

[1] Here and subsequently $\mathbf{1}$ denote the identity element in $\mathcal{M}$.

Lemma 4. *A system of equations*

$$\begin{cases} \lambda_1+\mu_1= & \lambda_0, \\ \lambda_2+\mu_2= & \mu_0, \\ \lambda_1^2+\lambda_2^2= & \lambda^2, \\ \mu_1^2+\mu_2^2= & \mu^2 \end{cases}$$

with respect to unknowns λ_i, μ_i $(i=1,2)$ *where*

$$\lambda_0^2+\mu_0^2=\lambda^2+\mu^2,$$

is solvable in $\mathbb{R}$. *In this case*

$$\lambda_1\mu_1+\lambda_2\mu_2=0.$$

A construction of the orthogonal vector measure

Now we examine some maximal commutative W^*-subalgebras in $\mathcal{N}$ that will useful for us. One such subalgebra is $\mathcal{M}\oplus\mathcal{M}$, the direct sum of two copies of $\mathcal{M}$,

$$\mathcal{M}\oplus\mathcal{M}=\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}:\ x,y\in\mathcal{M}\right\}.$$

Next, every pair (x,v) where

$$x\in\mathcal{M}^+,\ x\le\mathbf{1},\ \mathrm{rp}\,(x(\mathbf{1}-x))=\mathbf{1},\ v\in\mathcal{M}^{\mathrm{un}}, \tag{1}$$

can be associated with a maximal commutative W^*-subalgebra $\mathcal{N}_{x,v}$ in $\mathcal{N}$ described by the set of its projections

$$\mathcal{N}_{x,v}^{\mathrm{pr}}=\{p(x\pi_1,v,\pi_1)+p((\mathbf{1}-x)\pi_2,-v,\pi_2):\ \pi_i\in\mathcal{M}^{\mathrm{pr}},\ i=1,2\}.$$

It is easily seen that $\mathcal{N}_{x,v}$ is maximal. Note that $\mathcal{N}_{x,v}=\mathcal{N}_{\mathbf{1}-x,-v}$.

Let us to index the set of all such pairs, and associate to each $\gamma=(x,v)$ the set $\mathcal{N}_\gamma^{\mathrm{pr}}\equiv\mathcal{N}_{x,v}^{\mathrm{pr}}$ and associate to 0 the set $\mathcal{N}_0^{\mathrm{pr}}\equiv\{\pi_1\oplus\pi_2:\pi_i\in\mathcal{M}^{\mathrm{pr}}, i=1,2\}$. Then we totally order the set Γ of all indices γ, taking $0=\min\Gamma$.

It is known ([4, Proposition 1.18.1]) that a commutative W^*-algebra $\mathcal{M}$ may be realized as C^*-algebra $L^\infty(\Omega,\nu)$ of all essentially bounded locally ν-measurable functions on a localizable measure space (Ω,ν) (i. e. Ω is direct sum of finite measure spaces, see [5]). In this case, the Banach space $L^1(\Omega,\nu)$ is the predual of $L^\infty(\Omega,\nu)$: $L^1(\Omega,\nu)^*=L^\infty(\Omega,\nu)$. Now we shall identify $\mathcal{M}$ with $L^\infty(\Omega,\nu)$. In this case the characteristic functions

$$\pi(\omega)\equiv\chi_\pi(\omega)=\begin{cases} 1, & \text{if } \omega\in\pi, \\ 0, & \text{if } \omega\notin\pi, \end{cases} \qquad \pi\subset\Omega,$$

correspond to projections $\pi\in\mathcal{M}^{\mathrm{pr}}$. (The reader will note to his regret that we use the same letter π to designate three objects: a projection in $\mathcal{M}^{\mathrm{pr}}$, a

ν-measurable set in Ω, the characteristic function of this set.) By virtue of classical integration theory, for every measure $\sigma : L^\infty(\Omega,\nu)^{\rm pr} \to \mathbb{R}_+$ is determined uniquely a function $h \in L^1(\Omega,\nu),\ h \geq 0$, such that

$$\sigma(\pi) = \int \pi(\omega)h(\omega)\nu(d\omega) = \int\limits_{\pi} h(\omega)\nu(d\omega).$$

In this approach, the W^*-algebra $\mathcal{N}$ is realized as von Neumann algebra of 2×2-matrices $(x_{ij}),\ x_{ij} \in L^\infty(\Omega,\nu)$ acting on the orthogonal sum of two copies of Hilbert space $L^2(\Omega,\nu)$:

$$H = L^2(\Omega,\nu) \dotplus L^2(\Omega,\nu) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} :\ f,g \in L^2(\Omega,\nu) \right\}.$$

Next, let $m : \mathcal{N}^{\rm pr} \to \mathbb{R}_+$ be a given measure on projections on W^*-algebra $L^\infty(\Omega,\nu)\otimes M_2$. Let $0 \leq h_0, k_0 \in L^2(\Omega,\nu)$ such that

$$m(\pi_1 \oplus \pi_2) = \int\limits_{\pi_1} h_0^2(\omega)\nu(d\omega) + \int\limits_{\pi_2} k_0^2(\omega)\nu(d\omega), \quad \pi_i \in \mathcal{M}^{\rm pr}.$$

Similarly, there are $0 \leq h_\gamma, k_\gamma \in L^2(\Omega,\nu)$ such that

$$m(p(x\pi,v,\pi)) = \int\limits_{\pi} h_\gamma^2\, d\nu, \quad m(p((\mathbf{1}-x)\pi,-v,\pi)) = \int\limits_{\pi} k_\gamma^2\, d\nu, \quad \pi \in \mathcal{M}^{\rm pr}. \quad (2)$$

In addition, Lemma 2 and the Radon-Nykodim theorem give

$$h_\gamma^2(\omega) + k_\gamma^2(\omega) = h_0^2(\omega) + k_0^2(\omega) \quad \text{a. e.}$$

We will now state the main result of this paper.

Theorem 5. *Let $m : \mathcal{N}^{\rm pr} \to \mathbb{R}_+$ be a measure on projections in W^*-algebra $\mathcal{N}$ of type I_2. Then there exist a Hilbert space H and an orthogonal vector measure $\mu : \mathcal{N}^{\rm pr} \to H$ with property*

$$\|\mu(p)\|^2 = m(p), \qquad p \in \mathcal{N}^{\rm pr}.$$

Proof. Define an orthogonal vector measure μ on the set $[0] \equiv \mathcal{N}_0^{\rm pr}$ via

$$\mu(\pi_1 \oplus \pi_2) \equiv \begin{pmatrix} \pi_1 h_0 \\ \pi_2 k_0 \end{pmatrix}, \quad \pi_1, \pi_2 \in \mathcal{M}^{\rm pr}.$$

We next extend μ to an orthogonal vector measure on the set $[0,1]$ of all projections in the form

$$p = \pi_1 \oplus \pi_2 + p(x\pi_3, v, \pi_3) + p((\mathbf{1}-x)\pi_4, -v, \pi_4), \quad \pi_i \in \mathcal{M}^{\rm pr}, \qquad (3)$$

where (x, v) is a pair in (1) corresponding to index $1 \equiv \min(\Gamma \setminus \{0\})$. According to Lemma 2, it is possible to assume that $\pi_3\pi_4 = 0$. Thus,

$$\pi_1\pi_3 = \pi_1\pi_4 = \pi_2\pi_3 = \pi_2\pi_4 = \pi_3\pi_4 = 0.$$

One can easily see that the set $[0, 1]$ satisfies (iii) in Definition 2. In view of Lemma 3 there are real functions $0 \leq h_{1i}, k_{1i} \in L^2(\Omega, \nu), i = 1, 2$ such that equalities

$$h_{11}(\omega) + k_{11}(\omega) = h_0(\omega), \tag{4}$$
$$h_{12}(\omega) + k_{12}(\omega) = k_0(\omega), \tag{5}$$
$$h_{11}^2(\omega) + h_{12}^2(\omega) = h_1^2(\omega), \tag{6}$$
$$k_{11}^2(\omega) + k_{12}^2(\omega) = k_1^2(\omega), \tag{7}$$

$$h_{11}(\omega)k_{11}(\omega) + h_{12}(\omega)k_{12}(\omega) = 0. \tag{8}$$

hold a. e. (Here the functions h_1, k_1 in (2) correspond to index $\gamma = 1$.)

We now extend the function μ to projections in the form (3) putting

$$\mu(p) \equiv \begin{pmatrix} \pi_1 h_0 + \pi_3 h_{11} + \pi_4 k_{11} \\ \pi_2 k_0 + \pi_3 h_{12} + \pi_4 k_{12} \end{pmatrix}, \tag{27}$$

where h_{1i}, k_{1i} are solutions of the system (4) — (7). Direct computations with application (4) – (8) show that μ is the finitely additive orthogonal vector measure on $[0, 1]$.

Suppose (inductive hypothesis) that μ is extended to a finitely additive orthogonal vector measure on $[0, \gamma)$,

$$[0, \gamma) \equiv \{p_1 + ... + p_s : p_j \in \mathcal{N}_{\gamma_j}^{\mathrm{pr}}, p_j p_k = 0 \ (j \neq k), \ \gamma_j < \gamma\},$$

and $\gamma = (y, w)$. Let $0 \leq h_\gamma, k_\gamma \in L^2(\Omega, \nu)$ such that

$$m(p_\gamma) = \int_{\rho_1} h_\gamma^2 \, d\nu + \int_{\rho_2} k_\gamma^2 \, d\nu, \tag{9}$$

where

$$p_\gamma = p(y\rho_1, w, \rho_1) + p((\mathbf{1} - y)\rho_2, -w, \rho_2), \quad \rho_1\rho_2 \in \mathcal{M}^{\mathrm{pr}}. \tag{10}$$

Let

$$\mathcal{P} = \{\pi \in \mathcal{M}^{\mathrm{pr}} : \exists (x, v) < (y, w) \ (x\pi = y\pi, \ v\pi = w\pi)\},$$

and $(\pi_j)_{j \in J} \subset \mathcal{P}$ be a maximal set of pairwise orthogonal projections in $\mathcal{P}$ (it exists by Zorn's theorem). Define $\pi_0 \equiv \sum_j \pi_j (= \sup \mathcal{P})$.

With the above notations we have

$$p(y,w,\mathbf{1}) = p(y(\mathbf{1}-\pi_0),w,\mathbf{1}-\pi_0) + \sum_j p(y\pi_j,w,\pi_j)$$

$$= p(y(\mathbf{1}-\pi_0),w,\mathbf{1}-\pi_0) + \sum_j p(x_j\pi_j,v_j,\pi_j),$$

$$p(\mathbf{1}-y,-w,\mathbf{1}) = p((\mathbf{1}-y)(\mathbf{1}-\pi_0),-w,\mathbf{1}-\pi_0) + \sum_j p((\mathbf{1}-x_j)\pi_j,-v_j,\pi_j),$$

where $(x_j,v_j) < (y,w)$.

By inductive hypothesis there defined the functions $h_{j1},k_{j1},h_{j2},k_{j2} \in L^2(\Omega,\nu)$ satisfying equalities

$$\begin{aligned} h_{j1}(\omega)+k_{j1}(\omega) &= h_0(\omega),\\ h_{j2}(\omega)+k_{j2}(\omega) &= k_0(\omega),\\ h_{j1}^2(\omega)+h_{j2}^2(\omega) &= h_j^2(\omega),\\ k_{j1}^2(\omega)+k_{j2}^2(\omega) &= k_j^2(\omega),\end{aligned}$$

$$h_{j1}(\omega)k_{j1}(\omega)+h_{j2}(\omega)k_{j2}(\omega) = 0.$$

where the density functions h_j,k_j correspond to pairs (x_j,v_j) according to (2). We also find the functions $\widetilde{h}_{\gamma1},\widetilde{k}_{\gamma1},\widetilde{h}_{\gamma2},\widetilde{k}_{\gamma2} \in L^2(\Omega,\nu)$ that are solutions of equations

$$\begin{aligned} \widetilde{h}_{\gamma1}(\omega)+\widetilde{k}_{\gamma1}(\omega) &= h_0(\omega),\\ \widetilde{h}_{\gamma2}(\omega)+\widetilde{k}_{\gamma2}(\omega) &= k_0(\omega),\\ \widetilde{h}_{\gamma1}^2(\omega)+\widetilde{h}_{\gamma2}^2(\omega) &= h_\gamma^2(\omega),\\ \widetilde{k}_{\gamma1}^2(\omega)+\widetilde{k}_{\gamma1}^2(\omega) &= k_\gamma^2(\omega),\end{aligned}$$

where h_γ,h_γ are defined by (9). Therefore, there are defined the functions

$$h_{\gamma1}(\omega) \equiv (\mathbf{1}-\pi_0)\widetilde{h}_{\gamma1}(\omega) + \sum_j \pi_j(\omega)h_{j1}(\omega),$$

$$h_{\gamma2}(\omega) \equiv (\mathbf{1}-\pi_0)\widetilde{h}_{\gamma2}(\omega) + \sum_j \pi_j(\omega)h_{j2}(\omega),$$

$$k_{\gamma1}(\omega) \equiv (\mathbf{1}-\pi_0)^2(\omega)\widetilde{k}_{\gamma1}(\omega) + \sum_j \pi_j(\omega)k_{j1}(\omega),$$

$$k_{\gamma2}(\omega) \equiv (\mathbf{1}-\pi_0)\widetilde{k}_{\gamma2}(\omega) + \sum_j \pi_j(\omega)k_{j2}(\omega).$$

In this case

$$h_{\gamma 1}^2(\omega) + h_{\gamma 2}^2(\omega) = h_\gamma^2(\omega), \quad k_{\gamma 1}^2(\omega) + k_{\gamma 2}^2(\omega) = k_\gamma^2(\omega) \quad \text{a. e.},$$
$$h_{\gamma 1}(\omega) k_{\gamma 1}(\omega) + h_{\gamma 2}(\omega) k_{\gamma 2}(\omega) = 0 \quad \text{a. e.}$$

Now we put

$$\mu(p + p_\gamma) \equiv \mu(p) + \begin{pmatrix} \rho_1 h_{\gamma 1} + \rho_2 k_{\gamma 1} \\ \rho_1 h_{\gamma 2} + \rho_2 k_{\gamma 2} \end{pmatrix}.$$

where $p \in [0, \gamma)$ and p_γ is defined by (10). Again, direct computations show that μ is the finitely additive orthogonal vector measure on $[0, \gamma]$.

In view of Lemma 1 it follows that μ turned out extended to $\mathcal{N}^{\mathrm{pr}}$. Applying Lemma 3 we complete the proof. □

Corrolary 6. *Let $m : \mathcal{A}^{\mathrm{pr}} \to \mathbb{R}_+$ be a measure on projections in an arbitrary W^*-algebra $\mathcal{A}$. Then there exist complex Hilbert space H and an orthogonal vector measure $\mu : \mathcal{A}^{\mathrm{pr}} \to H$ such that*

$$\|\mu(p)\|^2 = m(p), \qquad p \in \mathcal{A}^{\mathrm{pr}}.$$

Proof. Because an orthogonal vector measure is uniquely defined by its restrictions to direct summands of a W^*-algebra, the statement follows by Theorem 5 and the proof of Theorem in [1]. □

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